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Universal Charge Distribution and Nonlocal Quantum Electrodynamics

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It is assumed that the electric charge of pointlike objects carries a nonlocality defined by a fundamental length and has a distribution over space. From physical requirements a unique form of charge distribution is found that in turn gives rise to a change in the Coulomb law at short distances and leads to a modification of the photon propagator. A nonlocal gauge transformation connected with extended charge is presented, which allows us to construct gauge-invariant nonlocal quantum electrodynamics free of ultraviolet divergences.

1. INTRODUCTION

The physical and space-time (or geometric) understanding of the origin of electric charge is an unsolved problem of modern physics. There are two approaches one can take concerning the fundamental nature of electric charge. First, from the physical point of view, the need with regard to extended, fundamental objects, as opposed to pointlike constituents, for an explanation of nonlocality (Aspect and Grangier, 1986; Penrose, 1989) in quantum mechanics and for the construction of a unified field theory (Green *et al.*, 1986; Polyakov, 1987) of all interactions including gravitation allows us to connect an electric charge (distribution) with a fundamental length (Namrai, 1986) (the size of extended objects) and to understand it as a topological defect (mode) on the string world-sheet [see Nanopoulos (1994) for discussion]. Second, on a deeper level, where the quantum fluctuations in the geometry of space are so great at small distances that even the topology fluctuates, forms "wormholes," and traps lines of force, as supposed by

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Wheeler (Misner *et al.*, 1973), the electric and nuclear charges give evidence for the presence of a submicroscale structure of space-time resembling a foamlike structure which is on the whole homogeneous. Thus, it seems, the fluctuations in the geometry make topological defects (or a multiply connected topology) which provide a natural description for the electric charge as electric lines of force trapped in the topology of a multiply connected space.

Our approach belongs to the first direction and is modest; we attempt to find a unique form of the electric charge distribution associated with the universal fundamental length and to construct nonlocal gauge-invariant quantum electrodynamics. In the given scheme, the Coulomb law is changed at short distances and the photon propagator turns out to be modified; the theory becomes nonlocal and finite on both the classical and quantum levels. The concrete aspect of our scheme is still in a somewhat primitive state by comparison with the sophistication of conventional "point" field theory.

In Section 2 we start with Poisson's equation for a pointlike charge as a basis for the construction of the local quantum field theory in which ultraviolet divergences are presented. In Section 3 we introduce the fundamental length into physics via a charge distribution over space, and infinitely sharp delta functions, involved in the definition of the pointlike charge distribution, are smeared out over the extension of fundamental objects. Poisson's equation for extended charge is obtained and a unique form of the charge density is found. The Coulomb law and the photon propagator are modified in consistency with the local theory of pointlike elementary constituents. Further, in order to construct nonlocal quantum electrodynamics (QED), we introduce in Section 4 a nonlocal gauge transformation induced by an extended electric charge distribution. The Efimov (1977) nonlocal S-matrix for QED is obtained in Section 5. Sections 6 and 7 deal with the regularization procedure and the gauge invariance for the S-matrix. Finally, in Sections 8 and 9, we study the simplest primitive Feynman diagrams and obtain a restriction on the value of the fundamental length in nonlocal QED. In the Appendix we give some mathematical computations.

2. A POINTLIKE CHARGE AND THE COULOMB LAW

Let us consider the pointlike charge e with the distribution $\rho(\mathbf{r}) = \delta(\mathbf{r})$ in space. Here $\delta(\mathbf{r})$ is the Dirac δ-function. Then the Poisson equation for its potential is

$$\Delta\varphi_c = -e\rho(\mathbf{r}) \tag{2.1}$$

the solution of which is the Coulomb Law

$$\varphi(\mathbf{r}) = \frac{e}{4\pi}\int \frac{d\mathbf{r}'\,\rho(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}'|} = \frac{e}{4\pi}\int \frac{d\mathbf{r}'\,\delta(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}'|} = \frac{e}{4\pi}\frac{1}{r} \tag{2.2}$$

It is well known that equation (2.1) and potential (2.2) are a part of classical and quantum theories of the electromagnetic "point" interaction. In both there exist divergences; for example, the self-energy of the pointlike classical charge

$$\begin{aligned} w &= \frac{e}{2}\int d^3r\,\rho(\mathbf{r})\varphi_c(\mathbf{r}) = \frac{1}{2}\int d^3r\,(\text{grad}\,\varphi_c(\mathbf{r}))^2 \\ &= \frac{1}{2}\int d^3r\,E^2 = \frac{e^2}{4\pi}\int_0^\infty \frac{dr}{r^2} \end{aligned} \tag{2.3}$$

goes to infinity and in QED the local Green function of the photon

$$\Delta_{\mu\nu}(x) = -g_{\mu\nu}\Delta(x) \tag{2.4}$$

where

$$\Delta(x) = \frac{1}{(2\pi)^4 i}\int d^4p\,e^{ipx}\tilde{\Delta}(p^2) \tag{2.5}$$

and $\tilde{\Delta}(p^2) = (-p^2 - i\epsilon)^{-1}$, $p^2 = p_0^2 - \mathbf{p}^2$, has a singularity at the point $x = 0$. In the Euclidean metric the expression (2.4) takes the form

$$\Delta_E(x) = \frac{1}{4\pi^2}\frac{1}{x_E^2}; \qquad x_E^2 = x_4^2 + \mathbf{x}^2 = -x^2, \quad x^2 = x_0^2 - \mathbf{x}^2 \tag{2.6}$$

It should be noted that in the static limit, the Fourier transform of (2.5) is related to the Coulomb potential (2.2) by

$$\varphi_c = \frac{e}{4\pi}\frac{1}{r} = \frac{e}{(2\pi)^3}\int d^3p\,e^{i\mathbf{pr}}\frac{1}{\mathbf{p}^2} \tag{2.7}$$

or

$$\frac{1}{\mathbf{p}^2} = \frac{1}{e}\int d^3r\,e^{-i\mathbf{pr}}\varphi_c(\mathbf{r}) \tag{2.8}$$

The latter should be understood as an improper integral (see the Appendix). As is seen above, the relationships (2.6) and (2.7) mean that the concept of a pointlike charge [its singular potential (2.2)] gives rise to the appearance of singularities in the local quantum field theory and vice versa.

3. A CHARGE DISTRIBUTION AND THE FUNDAMENTAL LENGTH

Recently, a majority of physicists have come to believe that in nature there exists a new fundamental constant of the dimension of length, alongside such constants as the velocity of light c and the Planck constant $\hbar$. This new universal constant should lead to a change in our concepts of the physical world, and, in particular, the concepts of space-time and locality (causality). Introducing such a constant into physics is needed for the understanding of the nonlocal nature of quantum physics (Namsrai, 1986) and for the description of extended, fundamental objects such as strings or superstrings (Green *et al.*, 1986) as a basis for unified theories of all interactions. Here we attempt to define this fundamental length by using physical characteristics of the electric charge. It may be that the very existence of a fundamental length is caused by an electric charge distribution over space. To realize this idea, we should smear out the infinitely sharp delta function involved in the definition of the idealized concept of a pointlike charge, by the following change

$$e\delta(\mathbf{r}) \Rightarrow e\rho_l(\mathbf{r}) \tag{3.1}$$

where a first consistent scheme is

$$\lim_{l\to 0} \rho_l(r) = \delta(\mathbf{r}) \tag{3.2}$$

Here the distribution $\rho_l(\mathbf{r})$ describes the extended electric charge due to the existence of the fundamental length. We assume that the charge distribution $\rho_l(\mathbf{r})$ has a universal characteristic and is independent of the concrete properties of elementary constituents (say, electrons, quarks, etc.) which carry the electric charge.

The change (3.1) leads to the "nonlocal" Poisson equation

$$\Delta\varphi_l(\mathbf{r}) = -e\rho_l(\mathbf{r}) \tag{3.3}$$

and its solution is

$$\varphi_l(\mathbf{r}) = \frac{e}{4\pi}\int \frac{d\mathbf{r}'\ \rho_l(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}'|} \tag{3.4}$$

This is a modified form of the Coulomb law at short distances. It is obvious that in accordance with the correspondence principle, the self-energy (2.3) and nonlocal photon propagator (2.4) are finite for $\rho_l(\mathbf{r})$. The nonlocal Coulomb potential is related to a nonlocal photon propagator by

$$\varphi_l(\mathbf{r}) = \frac{e}{(2\pi)^3}\int d^3p\ e^{i\mathbf{pr}}D(\mathbf{p}^2) \tag{3.5}$$

and

$$D_l(\mathbf{p}^2) = \frac{1}{e}\int d^3r\, e^{-i\mathbf{pr}}\varphi_l(\mathbf{r}) \tag{3.6}$$

in the static limit.

Now we attempt to find a unique form for the distribution $\rho_l(\mathbf{r})$. For this purpose, we consider the Green function of the photon field:

$$D_{\mu\nu}(x) = \frac{i}{(2\pi)^4}\, g_{\mu\nu} \int d^4p\, e^{ipx} D_l(p^2) \tag{3.7}$$

where its Fourier transform $D_l(p^2)$ in the static limit is given by formula (3.6):

$$D_l(\mathbf{p}^2) = \frac{1}{e}\int d^3r\, e^{-i\mathbf{pr}} \frac{e}{4\pi}\int d\mathbf{r}'\, \frac{\rho_l(\mathbf{r} - \mathbf{r}')}{|r'|} \tag{3.8}$$

in accordance with (3.4). Here

$$\rho_l(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3q\, e^{-i\mathbf{q}(\mathbf{r}-\mathbf{r}')}\tilde{\rho}_l(\mathbf{q}) \tag{3.9}$$

and

$$\frac{1}{|r'|} = \frac{1}{2\pi^2}\int d^3p\, e^{i\mathbf{pr}'}\, \frac{1}{\mathbf{p}^2} \tag{3.10}$$

where $\rho_l(\mathbf{q})$ is the Fourier transform of the charge density $\rho_l(\mathbf{r})$. It is easy to calculate that

$$D_l(\mathbf{p}^2) = \frac{1}{e}\frac{1}{\mathbf{p}^2}\,\tilde{\rho}_l(\mathbf{p}) \tag{3.11}$$

On the other hand, propagator (3.7) is defined by using some nonlocal photon field $A^l_\mu(x)$:

$$D_{\mu\nu}(x - y) = \langle 0|T\{A^l_\mu(x)A^l_\mu(y)\}|0\rangle \tag{3.12}$$

where the generalized field $A^l_\mu(x)$ should be determined from the interaction vertex:

$$\mathcal{L}_{\text{in}}(x) = eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x) \Rightarrow e\rho(x)A'_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x) \tag{3.13}$$

A modified form of the interaction vertex between electromagnetic and charged fields arises due to the charge distribution $\rho_l(\mathbf{x})$ and also (at the same time) a modification of the Coulomb law, which gives rise to a change of electromagnetic field

$$A_\mu(x) \Rightarrow A'_\mu(x) \Rightarrow \rho_l(x)A_\mu(x)$$

The factorization form (3.11) allows us to introduce the smeared-out electromagnetic field $A^l_\mu(x)$:

$$A^l_\mu(x) = \rho^2_l(x) \otimes A_\mu(x) = \int d^4y\, \rho^2_l(x-y)A_\mu(y)$$

and the generalized form of the interaction vertex

$$\mathcal{L}_{\rm in}(x) = A^l_\mu(x)j_\mu(x) \tag{3.14}$$

(their strict deduction will be given below), where

$$A^l_\mu(x) = \int d^4y\, \rho^2_l(x-y)A_\mu(y) \tag{3.15}$$

and $j_\mu(x) = e\overline{\psi}(x)\gamma^\mu\psi(x)$ is the local current of the fermion field $\psi(x)$. The quantity $\rho(x)$ in (3.12) and (3.14) is the generalized form of the charge density in the four-dimensional case. Generally speaking, $\rho_l(x)$ is a generalized function in the Minkowski space. In the Euclidean metric, $\rho(x_{\rm E})$ possesses a probability measure satisfying the condition

$$\int d^4x_{\rm E}\, \rho^2_l(x_{\rm E}) = 1 \tag{3.16}$$

With the choice (3.15), the photon propagator (3.12) turns out to be

$$D_{\mu\nu}(x-y) = \frac{i}{(2\pi)^4}\, g_{\mu\nu} \int d^4p\, e^{ip(x-y)}\, \frac{[\tilde{\rho}^2_l(p)]^2}{-p^2 - i\epsilon} \tag{3.17}$$

where $p^2 = p_0^2 - \mathbf{p}^2$ and $\tilde{\rho}^2_l(p)$ is the Fourier transform of the generalized charge density $\rho^2_l(x)$.

Thus, the expected charge distribution $\rho_l(\mathbf{r})$ should obey conditions (3.2)–(3.6) and the equality

$$\tilde{\rho}_l(\mathbf{p}) = [\tilde{\rho}^2_l(\mathbf{p}, p_0)]^2 |_{p_0 \Rightarrow 0} \tag{3.18}$$

i.e., in the static limit.

Theorem 1. The nonlocal charge distribution of the Gaussian form

$$\rho_l(\mathbf{r}) = \frac{1}{\pi^{3/2} l^3} \exp\left(-\frac{\mathbf{r}^2}{l^2}\right) \tag{3.19}$$

and its Euclidean extension

$$\rho^2_l(x_{\rm E}) = \frac{4}{\pi^2 l^4} \exp\left[-\frac{2}{l^2}(\mathbf{x}^2 + x_4^2)\right] \tag{3.20}$$

satisfy all the above conditions.

The proof is verified by direct calculations.

1. First of all, expressions (3.19) and (3.20) satisfy the normalization conditions

$$\int d^3r\,\rho_l(\mathbf{r}) = \frac{4\pi}{\pi^{3/2}l^3}\int_0^\infty dr\,r^2 e^{-r^2/l^2}$$

$$= \frac{4}{\pi^{1/2}l^3}\,\frac{1}{2[2(1/l^2)]}\,(\pi l^2)^{1/2} = 1$$

and

$$\int d^4x_\mathrm{E}\,\rho_l^2(x_\mathrm{E}) = \frac{4}{\pi^2 l^4}\,\pi^2\int_0^\infty du\,u e^{-(2u/l^2)}$$

$$= \frac{4}{l^4}\,\frac{1}{(2/l^2)^2}\,\Gamma(2) = 1$$

2. Their Fourier transforms are

$$\tilde\rho_l(\mathbf{p}) = \int d^3r\,e^{-i\mathbf{p}\mathbf{r}}\rho_l(\mathbf{p})$$

$$= \frac{1}{\pi^{3/2}l^3}\,\frac{4\pi}{p}\int_0^\infty dr\,r\sin(pr)e^{-r^2/l^2}$$

$$= \frac{4}{\sqrt{\pi}\,pl^3}\,\frac{p\sqrt{\pi}}{4(1/l^3)}\exp\left(-\frac{\mathbf{p}^2l^2}{4}\right) = \exp\left(-\frac{\mathbf{p}^2l^2}{4}\right) \qquad (3.21)$$

$$\tilde\rho_l^2(p_\mathrm{E}) = \int d^4x_\mathrm{E}\,e^{-ip_\mathrm{E}x_\mathrm{E}}\rho_l^2(x_\mathrm{E})$$

$$= \frac{4}{\pi^2 l^4}\,\frac{4\pi^2}{p}\int_0^\infty dx\,x^2 J_1(px)e^{-2x^2/l^2}$$

$$= \frac{16}{pl^4}\,\frac{p}{(4/l^2)^2}\exp\left[-\frac{p^2}{4(2/l^2)}\right] = \exp\left(-\frac{p_\mathrm{E}^2l^2}{8}\right) \qquad (3.22)$$

or in the pseudo-Euclidean metric $p_\mathrm{E}^2 = -p_\mathrm{pE}^2$

$$\tilde\rho_l^2(p) = \exp\left[\frac{p^2l^2}{8}\right], \qquad p^2 = p_0^2 - \mathbf{p}^2 \qquad (3.23)$$

From this, it is easily seen that

$$\rho_l^2(x) = \frac{1}{(2\pi)^4}\int d^4p\,e^{-ipx}\tilde\rho_l^2(p) = \exp\left(+\frac{\Box l^2}{8}\right)\delta^{(4)}(x) \qquad (3.24)$$

where

$$\Box = \frac{\partial^2}{\partial \mathbf{x}^2} - \frac{\partial^2}{\partial x_0^2}$$

Thus, the extended form of the charge distribution (3.24) in the Minkowski space is just the generalized distribution investigated in Efimov (1977, 1985). From the explicit formulas (3.21) and (3.23) one can see that the equality (3.18) holds automatically.

3. The pointlike charge and its local theory are obtained as a consequence of (3.2) with distribution (3.19), where

$$\lim_{l \Rightarrow 0} \frac{1}{\pi^{3/2} l^3} \exp\left(-\frac{\mathbf{r}^2}{l^2}\right) = \delta(\mathbf{r})$$

4. The modified Coulomb law (3.4) with (3.19) is

$$\varphi_l(r) = (e/4\pi r)\phi(r/e) \tag{3.25}$$

where $\phi(x)$ is the probability integral

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2}$$

5. By direct calculations (see the Appendix) it is easy to show that the Poisson equation (3.3) with (3.19) and (3.25) is valid identically.

6. Direct and inverse Fourier transforms (3.5) and (3.6) with the charge density (3.19) give the relationship between the propagator of the photon (3.11) in the static limit and the changed Coulomb law (3.25).

7. In our scheme, the self-energy of the extended charge is finite,

$$w_l = \frac{e}{2} \int d^3r\, \rho_l(\mathbf{r})\varphi_l(\mathbf{r}) = \frac{\alpha}{l} \frac{1}{(2\pi)^{1/2}}, \qquad \alpha = \frac{e^2}{4\pi}$$

and the photon propagator $D_{\mu\nu}(x) = -g_{\mu\nu}D(x)$,

$$D(x) = \frac{1}{(2\pi)^4 i} \int d^4p\, e^{ipx}\tilde{D}(P^2)$$

has no singularities at the point $x = 0$,

$$D(0) = \frac{1}{(2\pi)^4} 2\pi^2 \int_0^\infty du\, \frac{u}{2} \frac{e^{-ul^2/4}}{u}$$

$$= \frac{1}{16\pi^2} \int_0^\infty du\, e^{-ul^2/4} = \frac{1}{4\pi^2} \frac{1}{l^2}$$

Thus, we have obtained all the formulas necessary to construct nonlocal gauge-invariant quantum electrodynamics.

4. A NONLOCAL GAUGE TRANSFORMATION AND THE NONLOCAL ELECTROMAGNETIC INTERACTION

It is well known that interaction of charged fields $\varphi_j(x)$ with the electromagnetic field $A_\mu(x)$ is defined by the requirement of gauge invariance. This means that the physical content of the description of the electromagnetic field by using potentials $A_\mu(x)$ does not change under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu f(x) \tag{4.1}$$

since the electromagnetic tensor of the field

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x)$$

is invariant under the gauge transformation (4.1). It is usually assumed that the interaction of charged fields $\varphi_j(x)$ with the electromagnetic field $A_\mu(x)$ is invariant with respect to the group of the gauge transformations:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu f(x) \\ \varphi_j(x) &\to \varphi_j(x) \exp\{iq_j f(x)\} \\ \varphi_j^*(x) &\to \varphi_j^*(x) \exp\{-iq_j f(x)\} \end{aligned} \tag{4.2}$$

with an arbitrary function $f(x)$. Here q_j is the charge of the fields $\varphi_j(x)$. Invariance of the total Lagrangian $L(\varphi_j, \varphi_j^*, A_\mu)$ with respect to the gauge group (4.2) leads to charge conservation:

$$\partial_\mu j_\mu(x) = 0 \tag{4.3}$$

where

$$j_\mu(x) = i \sum_j q_j \left\{ \frac{\delta L}{\delta(\partial_\mu \varphi_j^*(x))} \varphi_j^*(x) - \frac{\delta L}{\delta(\partial_\mu \varphi_j(x))} \varphi_j(x) \right\} \tag{4.4}$$

It should be noted that the gauge transformation (4.2) means locality of the interaction of the electromagnetic field with the charged fields. A unique electromagnetic characteristic of the field $\varphi_j(x)$ is its charge q_j, which enters into the transformation (4.2). The explicit form of the interaction Lagrangian of the electromagnetic field with charged fields is defined by using the principle of minimality, which asserts that one gets the change

$$\begin{aligned} \partial_\mu \varphi_j(x) &\Rightarrow \{\partial_\mu - iq_j A_\mu(x)\} \varphi_j(x) \\ \partial_\mu \varphi_j^*(x) &\Rightarrow \{\partial_\mu + iq_j A_\mu(x)\} \varphi_j^*(x) \end{aligned} \tag{4.5}$$

under the action of the operator ∂_μ on the fields $\varphi_j(x)$ and $\varphi_j^*(x)$.

This is a generally accepted procedure which leads to the difficulties encountered in the local quantum field theory.

In order to generalize the theory in the case of a nonlocal electromagnetic interaction due to an extended electrical charge distribution we consider, first, the stationary picture when fields $\varphi_j(x)$ do not depend on the time variable. Then, instead of the gauge group transformation (4.2), the following nonlocal transformations are assumed:

$$\begin{aligned} A_0(\mathbf{x}) &\Rightarrow A_0(\mathbf{x}) \\ \mathbf{A}(\mathbf{x}) &\Rightarrow \mathbf{A}(\mathbf{x}) + \boldsymbol{\partial} f(\mathbf{x}) \\ \varphi_j(\mathbf{x}) &\Rightarrow \varphi_j(\mathbf{x}) \exp\left\{ iq_j \int d\mathbf{y}\, \rho_l^2(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \right\} \\ \varphi_j^*(\mathbf{x}) &\Rightarrow \varphi_j^*(\mathbf{x}) \exp\left\{ -iq_j \int d\mathbf{y}\, \rho_l^2(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \right\} \end{aligned} \tag{4.6}$$

where $\rho_l(\mathbf{x})$ is the charge distribution defined in (3.19). In this case, the current vector (4.4) takes the form

$$\mathbf{J}_l(\mathbf{x}) = \sum_j \int d\mathbf{y}\, \rho_l^2(\mathbf{x}-\mathbf{y}) \mathbf{j}_j(\mathbf{y}) \tag{4.7}$$

Here

$$\mathbf{j}_j(\mathbf{x}) = iq_j \left\{ \frac{\delta L}{\delta(\boldsymbol{\partial}\varphi_j^*(\mathbf{x}))} \varphi_j^*(\mathbf{x}) - \frac{\delta L}{\delta(\boldsymbol{\partial}\varphi_j(\mathbf{x}))} \varphi_j(\mathbf{x}) \right\}$$

is the local current, and therefore the electromagnetic interaction of the type of $\mathbf{A}(\mathbf{x})\mathbf{J}_l(\mathbf{x})$ means that the vector potential $\mathbf{A}(\mathbf{x})$ is associated with the local current $\mathbf{j}_j(\mathbf{x})$ of the jth charged particle through some spatial form factor $\rho_l^2(\mathbf{x}-\mathbf{y})$; that is just the charge distribution with the fundamental length l.

For the general case, when fields $\varphi_j(x)$ and $A_\mu(x)$ depend on the space-time variables $x^\mu = x^0, \mathbf{x}$, it is natural to use the following nonlocal gauge transformations:

$$\begin{aligned} A_\mu(x) &\Rightarrow A_\mu(x) + \partial_\mu f(x) \\ \varphi_j(x) &\Rightarrow \varphi_j(x) \exp\left\{ iq_j \int d^4y\, \rho_l^2(x-y) f(y) \right\} \\ \varphi_j^*(x) &\Rightarrow \varphi_j^*(x) \exp\left\{ -iq_j \int d^4y\, \rho_l^2(x-y) f(y) \right\} \end{aligned} \tag{4.8}$$

where $\rho_l^2(x)$ is the generalized charge distribution (3.24). These transformations entail the conservation of an extended electric current:

$$J_\mu^l(x) = i \sum_j q_j \int dy\, \rho_l^2(x - y)\left\{\frac{\delta L}{\delta(\partial_\mu \varphi_j^*(y))}\varphi_j^*(y) - \frac{\delta L}{\delta(\partial_\mu \varphi_j(y))}\varphi_j(y)\right\}$$

The local variant is obtained in the limit $l \to 0$ or $\rho_l^2(x - y)|_{l\to 0} = \delta(x - y)$.

In the nonlocal case, the usual procedure of the change (4.5) takes the form

$$\partial_\mu \varphi_j(x) \Rightarrow \left\{\partial_\mu - iq_j \int dy\, \rho_l^2(x - y)A_\mu(y)\right\}\varphi_j(x)$$

$$\partial_\mu \varphi_j^*(x) \Rightarrow \left\{\partial_\mu + iq_j \int dy\, \rho_l^2(x - y)A_\mu(y)\right\}\varphi_j^*(x) \tag{4.9}$$

We now turn to the construction of the nonlocal QED with the fundamental universal charge density (3.19) and the smeared-out photon field (3.15).

5. THE EFIMOV NONLOCAL *S*-MATRIX FOR NONLOCAL QED

As seen above, in the case of the interaction between electromagnetic and Dirac electron–positron fields, the total Lagrangian of classic fields has the form

$$L(x) = L_A^0(x) + L_e^0(x) + L_{\text{in}}(x) \tag{5.1}$$

where

$$L_A^0(x) = -\frac{1}{2}\,\partial_\nu A_\mu(x)\,\partial_\nu A_\mu(x) \tag{5.2}$$

$$L_e^0(x) = \bar\psi(x)(i\hat\partial - m)\psi(x), \qquad \hat\partial = \gamma^\mu \partial_\mu \tag{5.3}$$

$$L_{\text{in}}(x) = e\bar\psi(x)\hat A_\mu^l(x)\psi(x) \tag{5.4}$$

Here

$$A_\mu^l(x) \int dy\, \rho_l^2(x - y)A_\mu(y) \tag{5.5}$$

e is the electron charge and $\rho_l(x)$ is its distribution.

The Lagrangian of the free electromagnetic field $L_A^0(x)$ is written in a form in accordance with the Lorentz condition $\partial_\mu A_\mu(x) = 0$.

For QED the gauge group (4.8) acquires the form

$$A_\mu(x) \Rightarrow A_\mu(x) + \partial_\mu f(x)$$

$$\psi(x) \Rightarrow \psi(x) \exp\left\{ ie \int dy\, \rho_l^2(x - y) f(y) \right\} \tag{5.6}$$

$$\bar{\psi}(x) \Rightarrow \bar{\psi}(x) \exp\left\{ -ie \int dy\, \rho_l^2(x - y) f(y) \right\}$$

Formally, the S-matrix can be written in the form of T-products (Efimov, 1977, 1985),

$$S = T_\Lambda^\delta \exp\left\{ ie \int dx\, \bar{\psi}(x) \hat{A}^l(x) \psi(x) \right\} \tag{5.7}$$

where the symbol T_Λ^δ is the so-called Wick T-product or T^*-operation (e.g., Bogolubov and Shirkov, 1980) and the upper and lower indexes δ, Λ correspond to intermediate regularization procedures (defined below) which make finite all the matrix elements of the perturbation theory; δ, Λ are parameters of the regularization, and the limits $\Lambda \Rightarrow \infty$ and $\delta \Rightarrow 0$ mean a removal of the regularizations.

In order to construct the perturbation series for the S-matrix (5.7) by prescription of the usual local theory, it is necessary to change (in the Feynman diagrams)

$$\begin{aligned}
\Delta_{\mu\nu}(x - y) \Rightarrow D_{\mu\nu}(x - y) &= g_{\mu\nu} D(x - y) \\
&= \langle 0 | T[A_\mu^l(x) A_\nu^l(y)] | 0 \rangle \\
&= \int dy_1 \int dy_2\, \rho_l^2(x - y_1) \rho_l^2(y - y_2) \\
&\times \langle 0 | T\{A_\mu(y_1) A_\nu(y_2)\} | 0 \rangle \\
&= -g_{\mu\nu} \frac{1}{(2\pi)^4 i} \int dp\, \frac{[\tilde{\rho}_l^2(p^2 l^2)]^2}{-p^2 - i\epsilon} e^{-ip(x-y)}
\end{aligned} \tag{5.8}$$

and to keep the usual local fermion propagator

$$\begin{aligned}
S(x - y) &= \langle 0 | T\{\psi(x) \bar{\psi}(y)\} | 0 \rangle \\
&= \frac{1}{(2\pi)^4 i} \int dp\, \frac{1}{m - \hat{p} - i\epsilon} e^{-ip(x-y)}
\end{aligned} \tag{5.9}$$

The calculation of the matrix elements for the charged lepton loops will be undertaken using the following regularization procedure.

6. AN INTERMEDIATE REGULARIZATION PROCEDURE

The construction of the perturbation series for the S-matrix is possible only within the framework of a regularization procedure. In nonlocal quantum electrodynamics it is sufficient to regularize the nonlocal photon propagator and closed fermion loops. Thus, for the regularized photon propagator in momentum space one gets [see Efimov (1977, 1985) for details]

$$\tilde{D}^{\delta}(k^2) = \frac{l^2}{2i}\int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \, \frac{v(\xi)}{\sin \pi\xi} \, e^{\delta\xi^2}[l^2(-k^2 - i\epsilon)]^{\xi-1} \tag{6.1}$$

where we have used the Mellin representation

$$V(-p^2l^2) = \exp\left(\frac{p^2l^2}{4}\right) = \frac{1}{2i}\int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \, \frac{v(\xi)}{\sin \pi\xi} \, l^{2\xi}[-p^2]^{\xi} \tag{6.2}$$

$$v(x) = \frac{1}{\Gamma(1 + \xi)} \, 2^{-2\xi}, \qquad 0 < \alpha < 1$$

for the Fourier transform of the charge density $[\rho_l^2(x)]^2$.

For regularization of fermion propagators we will use the so-called Pauli–Villars gauge-invariant procedure. This means that causal fermion propagators are regularized not separately, but in closed spinor loops:

$$\sum_j c_j \, \mathrm{Sp}[\gamma S_{M_j}(x_1 - x_2)\gamma S_{M_j}(x_2 - x_3) \ldots] \tag{6.3}$$

where the coefficients c_j satisfy the following conditions (Efimov, 1977, 1985):

$$\begin{aligned} c_1 + c_2 + c_3 &= -1 \\ c_1\Lambda_1 + c_2\Lambda_2 + c_3\Lambda_3 &= -1 \\ c_1 \ln \Lambda_1 + c_2 \ln \Lambda_2 + c_3 \ln \Lambda_3 &= d \end{aligned} \tag{6.4}$$

$M_j^2 = m^2\Lambda_j$ and Λ_j ($j = 1, 2, 3$) are large dimensionless parameters ($\Lambda_j = \Lambda + \epsilon_j$, $\Lambda \gg 1$; $0 < \epsilon_j \ll 1$), and d is some finite number which must be chosen from the normalization condition of the physical charge of the electron.

Thus, the regularization introduced here makes it possible to pass to the Euclidean metric in any diagram of the perturbation theory.

We recall that the unique form factor (6.2) decreases only in the Euclidean direction, i.e., when $p^2 \Rightarrow -\infty$. Therefore we shall investigate the Feynman diagrams in the Euclidean momentum space. At the end of the calculations it is necessary to remove this intermediate regularization, i.e., to pass to the limit $\delta \Rightarrow 0$.

Moreover, spinor loops are finite in the limit $\Lambda \Rightarrow \infty$ in accordance with the conditions (6.4).

Finally, it should be noted that the limit

$$S = \lim_{\delta \Rightarrow 0} \lim_{\{\Lambda_j \Rightarrow \infty\}} S_\Lambda^\delta \tag{6.5}$$

exists and is obtained in such a way that the S-matrix is unitary and satisfies a macrocausality condition investigated by Efimov (1977, 1985).

In nonlocal quantum electrodynamics the interaction Lagrangian has formally the same form as in the local theory:

$$L_{\text{in}}(x) = e :\bar{\psi}(x)\hat{A}^l(x)\psi(x): + e(Z_l - 1) :\bar{\psi}(x)\hat{A}^l(x)\psi(x): - \delta m :\bar{\psi}(x)\psi(x): + (Z_2 - 1) :\bar{\psi}(x)(i\hat{\partial} - m)\psi(x): - (Z_3 - 1)\tfrac{1}{4} :F_{\mu\nu}(x)F_{\mu\nu}(x): \tag{6.6}$$

where the renormalized constants Z_1, Z_2, Z_3, and δm are finite and $Z_1 = Z_2$ in accordance with the Ward identity.

7. GAUGE INVARIANCE OF THE NONLOCAL S-MATRIX

A requirement of gauge invariance for the nonlocal S-matrix, i.e., invariance with respect to the transformation

$$A_\mu(x) \Rightarrow A_\mu(x) + \partial_\mu f(x) \tag{7.1}$$

with an arbitrary function $f(x)$, can be written in the form

$$\partial/\partial x_{1\mu_1} \cdots \partial/\partial x_{n\mu_n}(\delta^n S/\delta A_{\mu_1}(x_1) \cdots \delta A_{\mu_n}(x_n)) = 0 \tag{7.2}$$

where fermion operators of the electron field satisfy the free motion equation. For the proof of (7.2) it is sufficient to consider the case $n = 1$, i.e.,

$$\partial_\mu \frac{\delta S}{\delta A_\mu(x)} = 0 \tag{7.3}$$

Let us carry out a formal proof in terms of the representation

$$S = T \exp\left\{ i \int dx\, L_{\text{in}}(x) \right\} \tag{7.4}$$

Suppose that the representation (7.4) ensures the construction of the perturbation series with the causal functions (5.8) and the S-matrix is decomposed into series of normal products of field operators satisfying free motion equations. Thus, making use of (7.4), one gets

$$\frac{\delta S}{\delta A_\mu(x)} = iT\left\{\left(\frac{\delta}{\delta A_\mu(x)}\int dy\, L_{\text{in}}(y)\right)S\right\}$$
$$= i\int dx'\, \rho^2(x - x')T\left\{\left(\frac{\delta}{\delta A^l_\mu(x')}\int dy\, L_{\text{in}}(y)\right)S\right\}$$
$$= i\int dx'\, \rho^2(x - x')T\{e\bar{\psi}(x')\gamma_\mu\psi(x')S\} \quad (7.5)$$

Further, we take into account the following equalities:

$$T\{\psi(x)S\} = \psi(x)S + \int dy\, S(x - y)T\{ie\hat{A}^l(y)\psi(y)S\}$$
$$i\hat{\partial}\{\psi(x)S\} = T\{[m\psi(x) - ie\hat{A}^l(x)\psi(x)]S\}$$
$$i\partial_\mu\{\bar{\psi}(x)\gamma_\mu S\} = T\{[-m\bar{\psi}(x) + ie\bar{\psi}(x)\hat{A}^l(x)]S\} \quad (7.6)$$

These relations are valid if the perturbation theory is constructed in accordance with the Wick theorem with chronological pairing of the fermion operators (5.9), and the S-matrix depends on field operators satisfying free motion equations.

In terms of relations (7.6) one gets

$$\partial_\mu \frac{\delta S}{\partial A_\mu(x)} = \int dy\, \rho^2(x - y)\frac{\partial}{\partial y_\mu}\frac{\delta S}{\delta A^l_\mu(y)}$$
$$= \int dy\, \rho^2(x - y)$$
$$\times T\{[m(\bar{\psi}(y)\psi(y)) - ie\bar{\psi}(y)\hat{A}^l(x)\psi(y)$$
$$- m(\bar{\psi}(y)\psi(y)) + ie\bar{\psi}(y)\hat{A}^l(x)\psi(y)]S\} = 0 \quad (7.7)$$

So, the S-matrix is gauge invariant within the given formal consideration.

8. THE CALCULATION OF THE PRIMITIVE FEYNMAN DIAGRAMS

Let us calculate the matrix elements for the S-matrix corresponding to the primitive diagrams (Fig. 1) which are divergent in the usual local quantum electrodynamics.

8.1. The Diagram of Self-Energy

Here the corresponding term in the S-matrix can be written in the form (see Fig. 1a)

$$-i :\bar{\psi}(x)\Sigma_l(x-y)\psi(y): \tag{8.1}$$

where

$$\Sigma_l(x-y) = -ie^2\gamma_\mu S(x-y)\gamma_\mu D(x-y)$$

Passing to the momentum representation and making use of our regularization procedure δ which allows us to go to the Euclidean metric by using $k_0 \Rightarrow \exp(i\pi/2)k_4$, we get in the limit $\delta \Rightarrow 0$

$$\begin{aligned}\tilde{\Sigma}_l(p) &= \lim_{\delta\to 0}(-ie^2)\int dx\, e^{ipx}\gamma_\mu S(x)\gamma_\mu D^\delta(x)\\ &= \frac{e^2}{(2\pi)^4}\int dk_E \frac{V(k_E^2l^2)}{k_E^2}\gamma_\mu^{(E)}\frac{m-\hat{p}_E+\hat{k}_E}{m^2+(p_E-k_E)^2}\gamma_\mu^{(E)}\end{aligned} \tag{8.2}$$

Here $p_E = (-ip_0, \mathbf{p})$, $\gamma^{(E)} = (-i\gamma_0, \boldsymbol{\gamma})$, and $k_E = (k_4, \mathbf{k})$, so that

$$\begin{aligned} p_E k_E &= p_4k_4 + \mathbf{pk} = -ip_0k_4 + \mathbf{pk}\\ \hat{p}_E &= (p_E\gamma^{(E)}) = p_4\gamma_4 + \mathbf{p}\boldsymbol{\gamma} = -p_0\gamma_0 + \mathbf{p}\boldsymbol{\gamma}\\ &= -(p\gamma) = -\hat{\mathbf{p}}\\ \hat{k}_E &= k_4\gamma_4 + \mathbf{k}\boldsymbol{\gamma} = -i\gamma_0k_4 + \mathbf{k}\boldsymbol{\gamma}\\ \gamma_\mu^{(E)}\gamma_\nu^{(E)} + \gamma_\nu^{(E)}\gamma_\mu^{(E)} &= -2\delta_{\mu\nu} \qquad (\delta_{11} = \delta_{22} = \delta_{33} = \delta_{44} = 1)\\ \gamma_\mu^{(E)}\hat{p}_E\gamma_\mu^{(E)} &= -2\hat{p}_E, \qquad \hat{p}_E^2 = -p_E^2 = p^2\end{aligned} \tag{8.3}$$

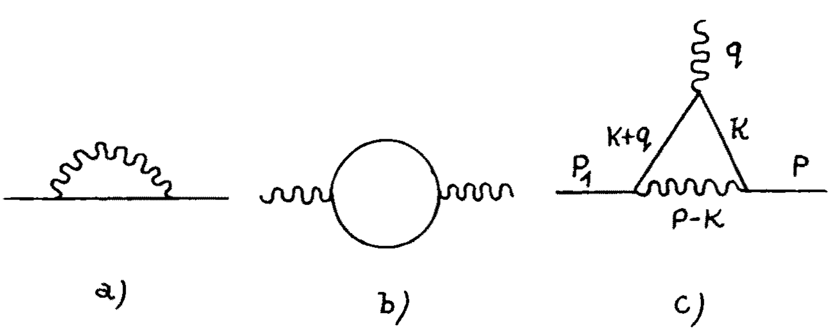

Fig. 1. The primitive Feynman diagrams in nonlocal QED.

Taking into account the Mellin representation (6.2) for the form factor $V(k_E^2 l^2)$ and after some calculations we have

$$\tilde{\Sigma}_l(p) = \frac{e^2}{8\pi}\frac{1}{2i}\int_{-\alpha+i\infty}^{-\alpha-i\infty} \frac{d\xi\, v(\xi)(m^2l^2)^\xi}{(\sin \pi\xi)^2\Gamma(1+\xi)} F(\xi, p) \tag{8.4}$$

where

$$F(\xi, p) = \frac{1}{\Gamma(1-\xi)}\int_0^1 du \left(\frac{1-u}{u}\right)^\xi \left(1 - \frac{p^2}{m^2}u\right)^\xi (2m - \hat{p}u) \tag{8.5}$$

is regular function in the half-plane Re $\xi > -1$. Assuming the value m^2l^2 to be small, one can obtain the following expression for the self-energy:

$$\begin{aligned}\tilde{\Sigma}_l(p) = {} & \frac{e^2}{8\pi^2}\int_0^1 du\,(2m - u\hat{p})\ln\left(1 - \frac{p^2}{m^2}u\right) - \frac{e^2}{16\pi^2} \\ & \times \left[\left(3\ln\frac{1}{m^2l^2} + 3v'(0) + 3\psi(1) + 1\right)\right. \\ & \left. + 4m^2l^2v(1)\left(\ln\frac{1}{m^2l^2} - \frac{v'(1)}{v(1)} - \frac{5}{12}\frac{p^2}{m^2}\right)\right] \\ & - \frac{e^2}{16\pi^2}(m - \hat{p})\left[\left(\ln\frac{1}{m^2l^2} - v'(0) + 1\right) - m^2l^2v(1)\frac{p^2}{3m^2}\right] \\ & + O((m^2l^2)^2)\end{aligned} \tag{8.6}$$

Let us calculate the correction to the electron mass,

$$\delta m = m_0 - m = -\tilde{\Sigma}(m) = \frac{3}{4\pi}\alpha\{\chi + O(1)\}m \tag{8.7}$$

where $\chi = \ln[1/(m^2l^2)]$.

As is seen above, the expression for $\tilde{\Sigma}(p)$ is consistent with the usual result in local quantum electrodynamics.

8.2. The Vacuum Polarization Diagram

The term of the scattering matrix corresponding to this diagram (Fig. 1b) has the form

$$-i :A_\mu(x)\Pi_{\mu\nu}(x-y)A_\nu(y): \tag{8.8}$$

where

$$\Pi_{\mu\nu}(x-y) = -ie^2 Sp\{\gamma_\mu S(x-y)\gamma_\nu S(y-x)\} \tag{8.9}$$

Let us use the proposed method of the regularization (6.3) and obtain in momentum space

$$\text{reg}\ \Pi_{\mu\nu}(x-y) = \frac{1}{(2\pi)^4}\int dp\, e^{-ip(x-y)}\ \text{reg}\ \tilde{\Pi}_{\mu\nu}(p) \tag{8.10}$$

Here

$$\text{reg}\ \Pi_{\mu\nu}(p) = \frac{e^2}{(2\pi)^4 i}\int dk \sum_{j=0}^{3} c_j Sp\left\{\gamma_\mu \frac{1}{M_j - \hat{k} - i\epsilon}\gamma_\nu \frac{1}{M_j - (\hat{k}-\hat{p}) - i\epsilon}\right\}$$

$$= (p_\mu p_\nu - g_{\mu\nu}p^2)\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)$$

$$\times\left\{\ln\left(1 - x(1-x)\frac{p^2}{m^2}\right) + \sum_{j=1}^{3} c_j \ln\left(\Lambda_j - x(1-x)\frac{p^2}{m^2}\right)\right\} \tag{8.11}$$

In virtue of the condition (6.4) one obtains in the limit $\Lambda \Rightarrow \infty$

$$\tilde{\Pi}_{\mu\nu}(p) = \lim_{\Lambda\to\infty}\lim_{\epsilon_j\to 0}\text{reg}\ \tilde{\Pi}_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu)\tilde{\Pi}(p^2) \tag{8.12}$$

where

$$\tilde{\Pi}(p^2) = \tilde{\Pi}_r(p^2) + \frac{e^2}{12\pi^2}d$$

and

$$\tilde{\Pi}_r(p^2) = \frac{e^2}{12\pi^2}p^2\int_{4m^2}^{\infty}\frac{d\rho^2}{\rho^2(\rho^2 - p^2 - i\epsilon)}\left(1 - \frac{4m^2}{\rho^2}\right)^{1/2}\left(1 + \frac{2m^2}{\rho^2}\right) \tag{8.13}$$

Thus, within the framework of our regularization procedure the polarization operator $\tilde{\Pi}(p^2)$ is finite upon the removal of the regularization and coincides with the renormalized expression in the usual local electrodynamics if we choose $d = 0$ for an arbitrary constant of the regularization. In this case $\tilde{\Pi}(p^2)$ is normalized by the condition

$$\Pi(0) = \tilde{\Pi}_r(0) = 0$$

This means that the constant d should define the renormalized electron charge and the choice $d = 0$ corresponds to the fact that, at least in second order of perturbation theory, the charge renormalization does not take place, i.e., the physical charge of the electron e coincides with the bare one e_0.

8.3. The Vertex Operator

Let us consider the diagram shown in Fig. 1c. Its matrix element is

$$ie\ :\bar{\psi}(x)\Gamma_\mu(x, z; y)\psi(z)A_\mu(y): \tag{8.14}$$

where we have introduced a vertex function of the third order

$$\Gamma_\mu(x, z; y) = ie^2\gamma_\nu S(x - y)\gamma_\mu S(y - z)\gamma_\nu D(x - z) \tag{8.15}$$

Taking into account momentum variables as shown in Fig. 1c and passing to the momentum representation, one can obtain in the Euclidean metric

$$\begin{aligned}\tilde{\Gamma}_\mu(p_1, p) &= \lim_{\delta \Rightarrow 0} ie^2 \int dy \int dz\, e^{ipz + iqy}\gamma_\nu S(y)\gamma_\mu S(z - y)\gamma_\nu D^\delta(z)\\ &= -\frac{e^2}{(2\pi)^4}\int \frac{dk_E\, V((p_E - k_E)^2 l^2)\gamma_\nu(m - \hat{k}_E - \hat{q}_E)\gamma_\mu(m - \hat{k}_E)\gamma_\nu}{(p_E - k_E)^2[m^2 + (k_E + q_E)^2](m^2 + k_E^2)}\end{aligned} \tag{8.16}$$

Let us carry out integration over the virtual momentum k_E in terms of the generalized Feynman parametrization:

$$\begin{aligned}&b_1^{-n_1} b_2^{-n_2} \cdots b_j^{-n_j}\\ &\quad = \frac{\Gamma(n_1 + \cdots + n_j)}{\Gamma(n_1) \cdots \Gamma(n_j)} \int_0^1 dx_1 \cdots \int_0^1 dx_j\, \delta\left(1 - \sum_{i=1}^{j} x_i\right)\\ &\quad\times x_1^{n_1 - 1} \cdots x_j^{n_j - 1}\left[\sum_{i=1}^{j} x_i b_i\right]^{-n_1 - \cdots - n_j}\end{aligned} \tag{8.17}$$

Again passing to the Minkowski metric, in accordance with the condition $(p_{iE}p_{jE}) = -(p_i p_j)$, one gets

$$\tilde{\Gamma}_\mu(p_1, p) = -\frac{e^2}{8\pi}\frac{1}{2i}\int_{-\alpha + i\infty}^{-\alpha - i\infty} \frac{v(\xi)(m^2 l^2)^\xi}{(\sin \pi\xi)^2\Gamma(1 + \xi)} F_\mu(\xi; p_1, p) \tag{8.18}$$

where

$$F_\mu(\xi; p_1, p) = \gamma_\mu F_1(\xi; p_1, p) + F_{2\mu}(\xi; p_1, p)$$

Here

$$F_1(\xi; p_1, p) = \frac{1}{\Gamma(1 - \xi)} \iiint_0^1 d\alpha\, d\beta\, d\gamma\, \delta(1 - \alpha - \beta - \gamma)\alpha^{-\xi} Q^\xi$$

$$F_2(\xi; p_1, p) = \frac{1}{\Gamma(-\xi)} \iiint_0^1 d\alpha\, d\beta\, d\gamma\, \delta(1 - \alpha - \beta - \gamma)\alpha^{-\xi} Q^{\xi-1}$$
$$\times \frac{1}{m^2} [m^2\gamma_\mu - 2mq_\mu + 4m(\beta q_\mu - \alpha p_\mu) \tag{8.19}$$
$$+ (\alpha\hat{p} - \beta\hat{q})\gamma_\mu\hat{q} + (\alpha\hat{p} - \beta\hat{q})\gamma_\mu(\alpha\hat{p} - \beta\hat{q})]$$
$$Q = \beta + \gamma - \alpha\gamma \frac{p^2}{m^2} - \beta\gamma \frac{q^2}{m^2} - \alpha\beta \frac{(p + q)^2}{m^2}$$

We now study the vertex function (8.18) for two cases: first, when $q = 0$ and p has an arbitrary value; second, when q is an arbitrary quantity and p, p_1 are situated on the mass shell. In the first case, assuming $q = 0$ in the formula (8.19) and after some calculations we have

$$F_\mu(\xi; p, p) = \frac{1}{\Gamma(1 - \xi)} \int_0^1 du \left(\frac{1 - u}{u}\right)^\xi \left(1 - u\frac{p^2}{m^2}\right)^\xi \left[u\gamma_\mu + \frac{2\xi u p_\mu(2m - u\hat{p})}{m^2 - up^2}\right] \tag{8.20}$$

Comparing the obtained formula with the expression (8.5) for the self-energy of the electron, it is easy to note that

$$F_\mu(\xi; p, p) = -\frac{\partial}{\partial p_\mu} F(\xi, p) \tag{8.21}$$

From this we can obtain a very important conclusion. In the nonlocal quantum electrodynamics constructed by using the concept of the extended charge density the Ward–Takahashi identity is valid,

$$\tilde{\Gamma}_\mu(p, p) = -\frac{\partial}{\partial p_\mu} \tilde{\Sigma}(p) \tag{8.22}$$

In the second case, one can put

$$\bar{u}(\mathbf{p}_1)\tilde{\Gamma}_\mu(p_1, p)u(\mathbf{p}) = \bar{u}(p_1)\Lambda_\mu(q)u(\mathbf{p}) \tag{8.23}$$

where $\bar{u}(\mathbf{p}_1)$ and $u(\mathbf{p})$ are solutions of the Dirac equations:

$$(\hat{p} - m)u(\mathbf{p}) = 0, \qquad \bar{u}(\mathbf{p}_1)(\hat{p}_1 - m) = 0$$

Substituting the vertex function (8.18) into (8.23) and after some transformations, we have

$$\bar{u}(\mathbf{p}_1)F_\mu(\xi; p_1, p)u(\mathbf{p}) = \bar{u}(p_1)\Lambda_\mu(\xi, q)u(\mathbf{p}) \tag{8.24}$$

Here

$$\Lambda_\mu(\xi, q) = \gamma_\mu f_1(\xi, q^2) + \frac{i}{2m}\sigma_{\mu\nu}q_\nu f_2(\xi, q^2)$$

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

$$f_j(\xi, q^2) = \frac{1}{\Gamma(1-\xi)}\iiint_0^1 d\alpha\, d\beta\, d\gamma$$

$$\times\, \delta(1 - \alpha - \beta - \gamma)\alpha^{-\xi}L^{\xi-1}g_j(\alpha, \beta, \gamma, q^2)$$

$$L = \lambda\alpha + (1-\alpha)^2 - \beta\gamma\frac{q^2}{m^2}$$

$$g_1(\alpha, \beta, \gamma, q^2) = [(1-\alpha)^2(1-\xi) + 2\alpha\xi] - [\beta\gamma + \xi(\alpha+\beta)(\alpha+\gamma)]\frac{q^2}{m^2}$$

$$g_2(\alpha, \beta, \gamma, q^2) = 2\alpha\xi(1-\alpha) \tag{8.25}$$

In order to avoid infrared divergences in the vertex function we have here introduced the parameter $\lambda = \mu_{\mathrm{ph}}^2/m^2$ taking into account the "mass" of the photon.

Finally, we obtain

$$\Lambda_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q_\nu F_2(q^2) \tag{8.26}$$

where

$$F_j(q^2) = -\frac{e^2}{8\pi}\frac{1}{2i}\int_{-\alpha+i\infty}^{-\alpha-i\infty}\frac{d\xi\, v(\xi)(m^2l^2)^\xi}{(\sin\pi\xi)^2\Gamma(1+\xi)}f_j(\xi, q^2)$$

It is easy to verify that the vertex function $\Lambda_\mu(q)$ satisfies the gauge-invariant condition

$$q_\mu\bar{u}(\mathbf{p}_1)\Lambda_\mu(q)u(\mathbf{p}) = 0 \tag{8.27}$$

Let us write the first terms of the decomposition for the functions $F_1(q^2)$ and $F_2(q^2)$ over two small parameters m^2l^2 and q^2/m^2:

$$F_1(q^2) = \frac{\alpha}{4\pi}\left[\chi - 2\sigma - v'(0) + \frac{9}{2} - 6c - 3m^2l^2v(1)\right] + \frac{\alpha}{2\pi}\frac{q^2}{m^2}$$

$$\times\left\{\frac{2}{3}\left(\frac{1}{2}\sigma - \frac{3}{8}\right) + \frac{m^2l^2}{3}\left[v(1)\left(-\chi + 2c - \frac{13}{6}\right) + v'(1)\right]\right\} \tag{8.28}$$

where $\sigma = \ln(m^2/\mu_{\text{ph}}^2)$, $c = 0{,}577215\ldots$ is the Euler constant, $\alpha = e^2/4\pi$, and χ is defined above in (8.7); and

$$F_2(q^2) = -\frac{\alpha}{2\pi}\left[1 - \frac{2}{3}\,v(1)m^2l^2\right] \tag{8.29}$$

The term in (8.28) independent of q^2 defines the renormalized constant Z_1 and is subject to renormalization. Other terms may be defined from experimental data which will be discussed below.

Now, following Efimov (1977, 1985), we consider the role of renormalized constants in the nonlocal QED, which we have introduced in the Lagrangian (6.6). The self-energy operator with the renormalized constants δm and Z_2 is written in the form

$$\tilde{\Sigma}_r(p) = \{A(p^2)m + B(p^2)\hat{p}\} + \delta m - (Z_2 - 1)(\hat{p} - m) \tag{8.30}$$

Here structure functions of the mass operator (8.4) and (8.5) are denoted by $A(p^2)$ and $B(p^2)$. Representation (8.30) is valid in any order of perturbation theory. Constants δm and Z_2 are chosen by the condition

$$\lim_{q\Rightarrow 0} \bar{u}(\mathbf{p})\,\frac{\tilde{\Sigma}_r(p+q)}{m - (\hat{p} + \hat{q})}\,u(\mathbf{p}) = 0 \tag{8.31}$$

where q is some four-vector such that $(pq) \neq 0$. The vector p lies on the mass shell, i.e., $p^2 = m^2$ and $\hat{p}u(\mathbf{p}) = mu(\mathbf{p})$. Substituting (8.30) into (8.31) and using the properties of the solutions of the Dirac equation

$$\bar{u}(\mathbf{p})\gamma_\mu u(\mathbf{p}) = \frac{p_\mu}{m}\,\bar{u}(\mathbf{p})u(\mathbf{p})$$

we obtain

$$\begin{aligned}\delta m &= -m(A(m^2 + B(m^2))\\ Z_2 - 1 &= B(m^2) + 2m^2(A'(m^2) + B'(m^2)) \qquad (8.32)\\ A'(m^2) &= \left.\frac{dA(p^2)}{dp^2}\right|_{p^2=m^2}\end{aligned}$$

Substituting the defined values of the renormalized constants into (8.30), one gets the following expression for the operator of mass:

$$\begin{aligned}\tilde{\Sigma}_r(p) = {} & m(A(p^2) - A(m^2)) + (B(p^2) - B(m^2))\hat{p}\\ & - 2m^2(A'(m^2) + B'(m^2))(\hat{p} - m)\end{aligned} \tag{8.33}$$

The vertex function with renormalized constant has the form

$$\tilde{\Gamma}_{r\mu}(p_1, p) = \tilde{\Gamma}_{\mu}(p_1, p) + (z_1 - 1)\gamma_{\mu} \tag{8.34}$$

The Ward–Takahashi identity (8.22) should be satisfied for the renormalized quantities $\tilde{\Sigma}_r(p)$ and $\tilde{\Gamma}_{r\mu}(p_1, p)$, and therefore

$$Z_2 = Z_1 \tag{8.35}$$

It should be noted again that all renormalized constants δm, Z_1, and Z_2 are finite and functions of the elementary length l in the nonlocal theory. Then δm and Z_2 are chosen from the normalization condition of the vertex operator (8.31) and the quantity Z_1 is defined from the Ward–Takahashi identity.

The renormalized operator of the vacuum polarization is written in the form

$$\Pi^r_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\nu p_\mu)\left[\tilde{\Pi}_r(p^2) + \frac{\alpha}{3\pi} d + Z_3 - 1\right] \tag{8.36}$$

Choosing $Z_3 = 1 - (\alpha/3\pi)d$, one obtains the normalization $\Pi_r(0) = 0$ for $\tilde{\Pi}^r_{\mu\nu}$. This condition entails that the charge e in the interaction Lagrangian is the physical observable charge and $\alpha = e^2/4\pi = 1/137$.

9. EXPERIMENTAL RESTRICTION ON THE VALUE OF THE FUNDAMENTAL LENGTH

Recently, no effect has been found experimentally which could not be dealt with within the local quantum electrodynamics. Tests of locality are usually performed by using very high precision experiments in atomic physics and in high-energy lepton–lepton scattering processes. In this section we calculate nonlocal corrections to the anomalous magnetic moments of leptons, the Lamb shift, and cross sections of the electromagnetic processes $e^-e^- \to e^-e^-$, $e^+e^- \to e^+e^-$, and $e^+e^- \to \mu^+\mu^-$, and obtain a restriction on the value of the fundamental length.

9.1. Corrections to the Anomalous Magnetic Moment (AMM) of Leptons

The nonlocal contribution to the AMM is defined from the vertex function $\Lambda_\mu(q)$ in (8.26) containing the term with $\sigma_{\mu\nu}q_\nu$, namely the formula (8.29) gives

$$\Delta\mu = \frac{\alpha}{2\pi}\left[1 - \frac{2}{3}\,\nu(1)m^2l^2\right] \tag{9.1}$$

and its first term corresponds to the Schwinger correction obtained in local

QED, where $\nu(1) = 1/4$ [see formula (6.2)]. At present, experimental values (Particle Data Group, 1998) of the AMM of the electron and muon are

$$\Delta\mu^{(e)}_{\exp} = 1.001159652193 \pm 0.000000000010$$
$$\Delta\mu^{(\mu)}_{\exp} = 1.001165923 \pm 0.000000008 \tag{9.2}$$

and are fully described by local QED. Comparing the correction (9.1) with the experimental errors in (9.2), one can obtain

$$l \leq 8.75 \times 10^{-15}\ \text{cm} \quad \text{for} \quad \Delta\mu^{(e)}_{\exp}$$
$$l \leq 1.2 \times 10^{-15}\ \text{cm} \quad \text{for} \quad \Delta\mu^{(\mu)}_{\exp} \tag{9.3}$$

9.2. The Lamb Shift

According to the standard calculation (Brodsky and Drell, 1970), the difference between energy levels $2S_{1/2}$ and $2P_{1/2}$ for the hydrogen atom due to the change of functions F_1 and F_2 in (8.26) is given by

$$\Delta E_l(2S_{1/2} - 2P_{1/2}) = \alpha^2 \mathrm{Ry}\left\{m^2 F_1'(0) - \frac{1}{2} F_2(0)\right\} \tag{9.4}$$

where $\mathrm{Ry} = m\alpha^2/2$ is the Rydberg constant.

Making use of the formulas (8.28) and (8.29), one obtains the following expression for the correction due to nonlocality of the electron charge:

$$\Delta E_l(2S_{1/2} - 2P_{1/2}) = \frac{\alpha^3}{6\pi} \mathrm{Ry}\cdot m^2 l^2 \left[\nu(1)\left(\ln m^2 l^2 + 2c - \frac{3}{2}\right) + \nu'(1)\right] \tag{9.5}$$

or with the function (6.2)

$$\Delta E_l(2S_{1/2} - 2P_{1/2}) = -\frac{\alpha^3}{6\pi} \mathrm{Ry}\, \frac{m^2 l^2}{4} \left[\ln \frac{1}{m^2 l^2} + \frac{5}{2} + 2\ln 2 - 3c\right] \tag{9.6}$$

$c = 0.577216$.

The experimental value of the Lamb shift is

$$(\Delta E)_{\exp} = (1057.912 \pm 0.011)\ \text{MHz/sec} \tag{9.7}$$

and is well explained by local QED (Brodsky and Drell, 1970).

Therefore

$$|\Delta E_l(2S_{1/2} - 2P_{1/2})| \leq 0.011\ \text{MHz/sec}$$

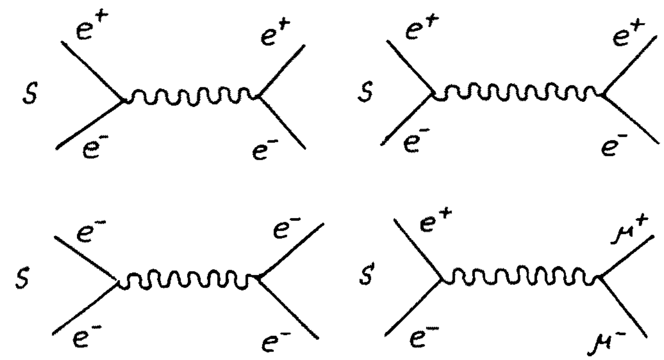

Fig. 2. Processes $e^-e^- \to e^-e^-$, $e^+e^- \to e^+e^-$, and $e^+e^- \Rightarrow \mu^+\mu^-$ in the low order of perturbation theory.

and substituting formula (9.6) into this equality, we get

$$l \leq 3 \times 10^{-13}\ \text{cm} \tag{9.8}$$

9.3. Electron Scatterings at High Energies

Stricter restrictions on the value of the fundamental length may be obtained from experiments on electron scattering at high energies. Electromagnetic processes of the type $e^-e^- \to e^-e^-$, $e^+e^- \to e^+e^-$, and $e^+e^- \Rightarrow \mu^+\mu^-$ are described by lower orders of perturbation theory (Fig. 2) even at the high energies attainable to date.

The ratio of cross sections calculated by local and nonlocal theories is

$$\frac{\sigma_{\text{nonlocal}}}{\sigma_{\text{local}}} = [V(-se^2)]^2 \sim 1 + 2v(1)sl^2 \tag{9.9}$$

where $s = (p_1 + p_2)^2 = (2E)^2 = W^2$, and $W = 2E$ is the total energy in the center-of-mass system. Estimation based on the formula (9.9) and experimental data (Bartel *et al.*, 1980; Berger *et al.*, 1980) is very simple and gives the restriction of the order

$$l \leq 10^{-16}\ \text{cm} \tag{9.10}$$

Finally, it should be noted that generally speaking restrictions (9.3), (9.8), and (9.10) imply that leptons as elementary constituents carrying extended electric charge are pointlike particles with radii smaller than 10^{-16} cm.

APPENDIX A

1. The Mellin representation (6.2) for an entire function is useful for calculating improper integrals. For example,

$$
\begin{aligned}
I_1 &= \int_0^\infty dx \, \frac{\sin ax}{x} \\
&= \lim_{\epsilon \to 0} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \, \frac{1}{\sin \pi\xi} \frac{a^{1+2\xi}}{\Gamma(2+2\xi)} \int_\epsilon^\infty dx \, x^{2\xi} \\
&= -\lim_{\epsilon \to 0} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \, \frac{1}{\sin \pi\xi} \frac{a^{1+2\xi}}{\Gamma(2+2\xi)} \frac{\epsilon^{1+2\xi}}{1+2\xi}, \qquad a > 0 \quad (\alpha > 0)
\end{aligned} \tag{A.1}
$$

Displacing the contour integration to the right and calculating the residues, one can easily see that the residue at the point $\xi = -1/2$ gives

$$I_1 = \pi/2 \tag{A.2}$$

in the limit $\epsilon \Rightarrow 0$.

Similarly, we have

$$
\begin{aligned}
I_2 &= \int_0^\infty dx \, \sin(ax^2) \\
&= -\lim_{\epsilon \to 0} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \, \frac{1}{\sin \pi\xi} \frac{1}{\Gamma(2+2\xi)} \frac{a^{1+2\xi}\epsilon^{4\xi+3}}{3+4\xi} = \frac{\sqrt{\pi}}{2\sqrt{2a}}
\end{aligned} \tag{A.3}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^\infty dx \, \sin ax \\
&= -\lim_{\epsilon \to 0} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \, \frac{1}{\sin \pi\xi} \frac{1}{\Gamma(2+2\xi)} \frac{a^{1+2\xi}\epsilon^{2\xi+2}}{2+2\xi} = \frac{1}{a}
\end{aligned} \tag{A.4}
$$

2. The modified Coulomb potential

$$\varphi_l(r) = \frac{e}{4\pi} \frac{1}{r} \phi\left(\frac{r}{l}\right) \tag{A.5}$$

satisfies the Poisson equation

$$\Delta\varphi_l(r) = \frac{-1}{\pi^{3/2} l^3} \exp\left(-\frac{r^2}{l^2}\right) \tag{A.6}$$

identically. Here we have used another representation for the probability integral

$$\phi(r/l) = 1 - (2/\sqrt{\pi}) \exp(-r^2/l^2)\, I_0 \tag{A.7}$$

$$I_0 = \int_0^\infty dt\, (t^2 + r^2)^{-1/2}(t/l) \exp(-t^2/l^2) \tag{A.8}$$

and the following integral forms:

$$i_1 = \int_0^\infty dt\, \frac{t}{l} (r^2 + t^2)^{-3/2} \exp\left(-\frac{t^2}{l^2}\right) = \frac{1}{l}\frac{1}{r} - \frac{2}{l^2} I_0 \tag{A.9}$$

$$i_2 = \int_0^\infty dt\, \frac{t}{l} (r^2 + t^2)^{-5/2} \exp\left(-\frac{t^2}{l^2}\right) = \frac{1}{3l}\frac{1}{r^3} - \frac{2}{3l^3}\frac{1}{r} + \frac{4}{3l^4} I_0 \tag{A.10}$$

which arise from the Laplacian $\Delta\varphi_l(r)$.

3. The equality

$$W = \frac{1}{2}\int d^3r\, \rho_l(\mathbf{r})\varphi_l(\mathbf{r}) = \frac{1}{2}\int d^3r\, \mathbf{E}_l^2(\mathbf{r}) \tag{A.11}$$

$$\mathbf{E} = -\nabla\left(\frac{e}{4\pi}\frac{1}{r}\phi\left(\frac{r}{l}\right)\right)$$

for the self-energy of the extended electric charge gives following useful formulas:

$$\int_0^\infty dy\, \frac{\phi^2(y)}{y^2} = \frac{\sqrt{2}}{\sqrt{\pi}}\left[\sqrt{2}\ln\frac{1+\sqrt{2}}{\sqrt{2}-1} - 1\right] \tag{A.12}$$

or

$$\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \frac{[2(n_1+n_2)-1]!!}{(2n_1+1)!!\,(2n_2+1)!!} 2^{-(n_1+n_2)} = \sqrt{2}\ln\frac{1+\sqrt{2}}{\sqrt{2}-1} - 1 \tag{A.13}$$

The latter means that an explicit sum of the symmetric double series (A.13) exists.

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